A NOTE ON ANCILLARITY AND INDEPENDENCE VIA MODEL-PRESERVING TRANSFORMATIONS.

by

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Summary

Let G be a compact group of transformations which preserves a family $\mathcal{P}$ of probability measures. Let T be a statistic such that its space $\mathcal{T}$ is a G-space. If T is equivariant and G is transitive on $\mathcal{T}$ then T is ancillary for $\mathcal{P}$ and distributed independently of a maximal G-invariant statistic T^* which also happens to be sufficient for $\mathcal{P}$. This result in then extended to a group G satisfying the Hunt-Stein Condition.

1. Introduction

This note originated from the following question Professor M. Eaton asked me. "Is $\underset{\sim}{X}/\|\underset{\sim}{X}\|$ ancillary and distributed independently of $\|\underset{\sim}{X}\|$ when the distribution of $\underset{\sim}{X}$ on R^n is preserved under orthogonal linear transformations?" The answer to this is known to be in the affirmative when the distribution of $\underset{\sim}{X}$ is also normal. The following fact seems to be related to this issue. If the distribution of $\underset{\sim}{X}$ on R^n is continuous and preserved under all permutations of its coordinates then the set of order statistics is sufficient and distributed independently of the ranks. We shall recall the standard proof of this result in order to answer Eaton's question in the affirmative in a broader framework.

Consider a family $\mathcal{P}$ of probability measures on a measure space $(\mathcal{X},\mathcal{A})$. Let G be a (one-to-one bimeasurable) group of transformations on $\mathcal{X}$ to $\mathcal{X}$. We say that $\mathcal{P}$ is preserved under G, if for all $A \in \mathcal{A}$, $g \in G$, $P \in \mathcal{P}$

(1) $$P(A) = P(g^{-1}A) .$$

Let $\mathcal{A}^*$ be the sub-σ-field of all G-invariant measurable sets. Suppose (for simplicity) that T^* is a maximal G-invariant statistic.

We shall find conditions for which (i) T^* is sufficient for $\mathcal{P}$, (ii) a given statistic T is ancillary for $\mathcal{P}$, and (iii) T and T^* are independently distributed.

2. The Main Result.

We assume that the group G is locally compact with a given topology and $\mathcal{G}$ is the collection of all its Borel sets. We assume that the transformation $(g,x) \to g(x)$ is jointly measurable in the product space $(G \times \mathfrak{X}, \mathcal{G} \times \mathcal{A})$.

Consider a statistic T with the associated space $(\mathcal{T},\mathcal{B})$. We shall use the following two conditions in proving our results.

Condition 1. For every $g \in G$

$$T(x_1) = T(x_2) \Rightarrow T(gx_1) = T(gx_2) .$$

Condition 2. Given any x_1 and x_2 in $\mathfrak{X}$ there exists a g (depending on x_1 and x_2) in G such that

$$T(x_1) = T(gx_2) .$$

Suppose that Condition 1 holds. Then every $g \in G$ induces a transformation $\tilde{g}$ on $\mathcal{T}$ to $\mathcal{T}$ given by

$$\tilde{g}\, T(x) = T(gx) .$$

Let $\tilde{G}$ be the collection of all such $\tilde{g}$'s . Then Condition II simply means that

Condition 3. $\tilde{G}$ is trnasitive on $\mathcal{T}$.

Theorem 1. Suppose that a family $\mathcal{P}$ of probability measures is preserved under a compact transformation group G . Let T^* be a maximal G-invariant statistic and T be a statistic satisfying Conditions 1 and 2. Then (i) T^* is sufficient for $\mathcal{P}$, (ii) T is ancillary for $\mathcal{P}$, and (iii) T and T^* are independently distributed .

Proof. There exists a left-invariant probability measure ν on $(G,\mathcal{G})$. For $A \in \mathcal{A}$, $A^* \in \mathcal{A}^*$, $P \in \mathcal{P}$, $g \in G$

(2.1) $$P(A \cap A^*) = P(gA \cap A^*)$$

Hence

(2.2) $$P(A \cap A^*) = \int_G P(gA \cap A^*)d\nu(g)$$

Define

(2.3) $$\tilde{A} = \{(x,g) \in \mathcal{X} \times G\colon g^{-1}x \in A\} ,$$

and

(2.4) $$\tilde{A}_x = \{g \in G\colon g^{-1}x \in A\} .$$

Applying Fubini's Theorem to the left-hand side of (2.2) we get

(2.5) $$P(A \cap A^*) = \int_{A^*} \nu(\tilde{A}_x)dP(x) .$$

Since for $h \in G$

(2.6) $$\tilde{A}_{hx} = h\,\tilde{A}_x ,$$

we find that $\nu(\tilde{A}_x)$ is G-invariant. Hence T^* is sufficient for $\mathcal{P}$.

Now choose and fix any arbitrary $x_0 \in \mathcal{X}$. By Condition II there exists $g_x \in G$ such that

(2.7) $$T(x) = T(g_x^{-1}x_0) .$$

By Condition I

(2.8) $$T(g^{-1}x) = T(g^{-1}g_x^{-1}x_0)$$

for every $g \in G$. For $B \in \mathcal{B}$ write $A = T^{-1}(B)$. Then

$$\tilde{A}_x = \{g \in G\colon g^{-1}x \in T^{-1}B\}$$

$$= \{g \in G\colon g^{-1}g_x^{-1}x_0 \in T^{-1}B\} .$$

(2.9) $$= g_x^{-1}\tilde{A}_{x_0} \quad .$$

Since ν is left-invariant

(2.10) $$\nu(\tilde{A}_x) = \nu(\tilde{A}_{x_0}) \quad .$$

This shows that the conditional probability of $T \in B$ given $\mathcal{G}^*$ is a constant free from $T^*(x)$. The results (ii) and (iii) now follow.

Note that the conditions 1 and 2 are implied also by the following:

Condition 4. G^* is a transitive group of (one-to-one onto bimeasurable) transformations on $\mathcal{X}$ to $\mathcal{X}$ such that G^* can be factored directly as $G^* = G \times H$, and T is a maximal H-invariant statistic.

Theorem 1 (i) was proved by Farrell [4].

Remark 1. Usually bounded completeness and sufficiency of T^* is used to show that T is ancillary is equivalent to independence of T and T^* [6]. Theorem 1 would be useful in many cases where the above result does not hold, or it is difficult to verify the bounded completeness of T^*. To apply Theorem 1 one needs only to check the simple conditions 1 and 2. Moreover, Theorem 1 generalizes many well-known results without assuming the existence of density functions. We shall illustrate these remarks by some examples.

Remark 2. Suppose both $\mathcal{X}$ and $\mathcal{J}$ are G-spaces. Then the mapping defined by (8) is called equivariant. In that case, let $C \subset \mathcal{X}$ be a cross-section of $\mathcal{X} \to \mathcal{X}/G$. Let φ: $C \to \mathcal{J}$ be a map such that the isotropy group $G_c \subset G_{\varphi(c)}$ for all $c \in C$. Then there is a unique extension of φ to an equivariant map T: $\mathcal{X} \to \mathcal{J}$ [3].

Remark 3. It was proved by Farrell [4] and later by Basu [1] that $\mathcal{G}^*$ is sufficient when G is countable. In the dominated case Theorem 1 (i) can be proved more easily [1,2].

3. Examples

(a) Let $\mathfrak{X} = \{x \in R^n : x \neq 0\}$, and $\mathcal{C}$ be the class of all Borel subsets of $\mathfrak{X}$. Let G be the group of all orthogonal linear transformations. Condition 3 can be verified easily. Then Theorem 1 holds with

$$T^*(x) = \|x\|, \quad T(x) = x/\|x\| .$$

In fact, the condition (4) holds with $H = \{h_\tau : \tau > 0\}$, $h_\tau x = \tau x$.

(ii) Consider the special case of the above example when $n = 1$. In that case G is the group of all sign transformations, and $T^*(x) = |x|$, $T(x) = \text{sign}(x)$. It may be of interest to note the following example:

$$P(X = 1) = P(X = -1) = \theta/2 , \; P(X = 2) = P(X = -2) = (1-\theta)/2 ,$$

where $0 \leq \theta \leq 1$. Let

$$T_1(x) = \begin{cases} 1, & \text{if } x = 1, -2 \\ 0, & \text{if } x = -1, 2 . \end{cases}$$

Then T_1 is not a function of $\text{sign}(x)$; but T_1 is ancillary and distributed independently of $|x|$.

(iii) Let S be a symmetric $n \times n$ random matrix. Consider a family $\mathcal{P}$ of distributions of S which is preserved under orthogonal linear transformations, i.e., S and gSg' have the same distribution for every $n \times n$ orthogonal matrix g. Note that S can be decomposed (uniquely with some conventions) as

$$S = L_S D_S L_S' ,$$

where L_S is orthogonal and D_S is diagonal with its diagonal elements as the eigenvalues of S. Let T^* be the vector consisting of the diagonal elements of S, and $T = L_S$. Since the condition 3 holds in this case, Theorem 1 also holds with the above T and T^*.

(iv) Let $\mathcal{X} = \{x \in R^n: x \neq 0\}$ and $\mathcal{G}$ be the class of all Borel subsets of $\mathcal{X}$. Suppose $\mathcal{P}$ is preserved by the Haggstorm subgroup of $n \times n$ orthogonal matrices (i.e., $n \times n$ orthogonal matrices with unit row sums and unit column sums). An example of such a $\mathcal{P}$ is $\mathcal{N}_n(\mu \underset{\sim}{1}_n, \Sigma)$, where $\Sigma = [\sigma_{ij}]$ is positive definite with $\sigma_{ij} = \rho\sigma^2$ for $i \neq j$ and $\sigma_{ij} = \sigma^2$ for $i = j$, and $\underset{\sim}{1}_n$ is the $n \times 1$ vector with all 1's. Then Theorem 1 holds with,

$$T^*(X) = (\bar{X} = \sum_{i=1}^{n} X_i/n\ , \sum_{i=1}^{n} X_i^2)\ ,$$

$$T(X) = (X_1-\bar{X}, \ldots, X_n-\bar{X})/[\sum_{i=1}^{n} (X_i-\bar{X})^2]^{\frac{1}{2}}\ .$$

(v) Let R be the additive group of reals with the usual topology and Z be the subgroup of all integers. Then the quotient group $G = R/Z$ is compact with respect to the quotient topology [7]. It can be seen that G is isomorphic to the group $G_c = \{g_c: 0 \leq x < 1\}$ with $g_{c_1} + g_{c_2} = g_c$, where $c = c_1 + c_2 \pmod 1$.

Let $\mathcal{P}$ be the family of uniform distributions on $[0,\theta)$, with θ being a positive integer. Consider

$$g_c x = [x] + \{x - [x] + c \pmod 1\}\ ,$$

where $[x]$ is the integer part of x (see [1]). Then Theorem 1 holds with

$$T^*(x) = [x],\ T(x) = x-[x]\ .$$

4. An Extension of Theorem 1.

Next we shall use Hunt-Stein's condition [6] given below to extend Theorem 1.

Condition 5. There exists a sequence $\{\nu_n\}$ of probability measures on $(G,\mathcal{G})$ such that for any $g \in G$, $B \in \mathcal{G}$

$$\lim_{n\to\infty} |\nu_n(gB) - \nu_n(B)| = 0 \quad .$$

Theorem 2. Theorem 1 holds if the compactness of G is replaced by Condition 5, and $\mathfrak{X}$ is Euclidean.

Proof. Proceeding as before we get

$$(4.1) \qquad P(A \cap A^*) = \int_{A^*} \nu_n(\tilde{A}_x)\, dP(x)$$

for $A \in \mathcal{A}$, $A^* \in \mathcal{A}^*$ and $\tilde{A}_x$ given by (2.4). From (2.9) we get

$$(4.2) \qquad \nu_n(\tilde{A}_x) = \nu_n(g_x^{-1}\tilde{A}_{x_0}) \quad ,$$

where $A = T^{-1}B$, $B \in \mathcal{B}$.

Condition 5 implies (see [6], p. 337) that for any fixed x and $g \in G$

$$(4.3) \qquad \nu_n(g\tilde{A}_x) - \nu_n(\tilde{A}_x) \to 0$$

as $n \to \infty$. Hence

$$(4.4) \qquad \nu_n(\tilde{A}_x) - \nu_n(\tilde{A}_{x_0}) \to 0$$

as $n \to \infty$. By Lebesgue dominated convergence theorem

$$(4.5) \qquad \int_{A^*} [\nu_n(\tilde{A}_x) - \nu_n(\tilde{A}_{x_0})]\, dP(x) \to 0 \; .$$

Thus from (4.1) we get

(4.6) $$P(A \cap A^*) = P(A^*) \lim_{n\to\infty} \nu_n(\widetilde{A}_{x_0}) \quad .$$

Clearly now

(4.7) $$P(A) = \lim_{n\to\infty} \nu_n(\widetilde{A}_{x_0}) \quad .$$

This shows $P(A)$ is a constant and $P(A \cap A^*) = P(A) \cdot P(A^*)$. The results (ii) and (iii) now follow. The fact that $\mathfrak{A}^*$ is sufficient for $\mathcal{P}$ can be proved easily proceeding exactly as in Lehmann ([6], pp. 336-7).

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Two generalizations of Brunn-Minkowski inequality for convex sets are presented along with their direct and simple proofs. Literature in this area is reviewed. The connection between Anderson's inequality and these inequalities are discussed. Several references for statistical applications are cited.